M-Matrix Inverse problem for distance-regular graphs

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Preliminaries

Matrices with **non–positive** off–diagonal and **non–negative** diagonal entries

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If $k \geq \rho(A)$, then L is called an ***M*–matrix**

Preliminaries

Symmetric, irreducible M–matrices

- **non–singular** ⇢ discrete Dirichlet problem ⇢ its inverse corresponds with the Green operator associated with the boundary value problem.

- **singular** ⇢ discrete Poisson equation ⇢ its Moore–Penrose inverse corresponds with the Green operator too.

Preliminaries

X is a **generalized inverse** of matrix A if

- exists for a class of matrices larger than the class of singular matrices
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Illustration

Given a system of linear equations $Ax = b$, if X is any matrix such that $AXA = A$, then

$$Ax = b \text{ has a solution } \Leftrightarrow AXb = b,$$

and the general solution is

$$x = Xb + (I - XA)y$$

where y is arbitrary.

Preliminaries

Generalized inverses

For every finite matrix A there is a unique matrix X satisfying the Penrose equations

$$\begin{aligned} AXA &= A, && (1)\\ XAX &= X, && (2)\\ (AX)^* &= AX, && (3)\\ (XA)^* &= XA, && (4)\end{aligned}$$

where A^* denotes the conjugate transpose of A.
Matrix X is commonly known as the **Moore–Penrose inverse**, and is denoted by $A^\dagger$.

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Known results

- An irreducible and non–singular M–matrix has inverse with all entries positive.

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- An irreducible and non–singular M–matrix has inverse with all entries positive.

- An irreducible and singular M–matrix has **a** generalized inverse which is non–negative.

Main question

When the Moore–Penrose inverse of a symmetric, singular and irreducible M–matrix is itself an M–matrix?

Notation

- **Finite network**: $\Gamma = (V, E, c)$
- The **conductance**: $c\colon V \times V \longrightarrow [0, +\infty)$ such that $c(x, x) = 0$ for any $x \in V$ and $c(x, y) > 0$ iff $x \sim y$.
- The **combinatorial Laplacian** of Γ

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y) \Big(u(x) - u(y) \Big), \quad x \in V.$$

⋇ $\mathcal{L}$ is a positive semi–definite self–adjoint operator.
⋇ $\mathcal{L}$ has 0 as its lowest eigenvalue whose associated eigenfunctions are constant.
⋇ $\mathcal{L}$ can be interpreted as an irreducible, symmetric, diagonally dominant and singular M–matrix, L.

Poisson equation. Operators L and G

The *Poisson equation*

$$\mathcal{L}(u) = f \text{ on } V \text{ has solution} \Leftrightarrow \sum_{x\in V} f(x) = 0.$$

$$\exists! u \in \mathcal{C}(V) \text{ such that } \sum_{x\in V} u(x) = 0.$$

The *Green operator* is the linear operator $\mathcal{G} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to any $f \in \mathcal{C}(V)$ the unique solution of the Poisson equation with data $f - \frac{1}{n}\sum\limits_{x\in V} f(x)$ such that $\sum\limits_{x\in V} u(x) = 0$.

⋇ $\mathcal{G}$ is a positive semi–definite self–adjoint operator.

⋇ $\mathcal{G}$ has 0 as its lowest eigenvalue whose associated eigenfunctions are constant.

⋇ If $\mathcal{P}$ denotes the projection on the subspace of constant functions then, $\mathcal{L} \circ \mathcal{G} = \mathcal{G} \circ \mathcal{L} = \mathcal{I} - \mathcal{P}$.

Green function

The *Green function* $G : V \times V \longrightarrow \mathbb{R}$ is defined by $G(x, y) = \mathcal{G}(\varepsilon_y)(x)$, where ε_y is the Dirac function at y.

Interpreting $\mathcal{G}$ or G as a matrix:
⋇ G is the Moore–Penrose inverse of L
⋇ G is an M–matrix if, and only if $G(x, y) \leq 0$ for any $x, y \in V$ with $x \neq y$.

The equilibrium measure and the capacity

There exists $\nu^x \in \mathcal{C}(V)$ such that $\begin{cases} \nu^x(x) = & 0 \\ \nu^x(y) > & 0 \quad y \neq x \end{cases}$
and verifying

$$\mathcal{L}(\nu^x) = 1 - n\varepsilon_x \quad \text{on } V.$$

We call ν^x the **equilibrium measure** of $V \setminus \{x\}$
The **capacity** is the function cap $\in \mathcal{C}(V)$ given by

$$\text{cap}(x) = \sum_{y \in V} \nu^x(y).$$

Question

When the Moore–Penrose inverse of the combinatorial Laplacian matrix of a distance–regular graph is an M–matrix?

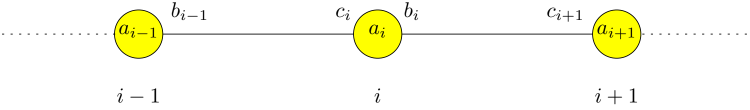
Distance–regular graphs

Let Γ be a distance–regular graph with intersection array

$$\iota(\Gamma) = \{b_0, b_1, \ldots, b_{D-1}; c_1, \ldots, c_D\},$$

Γ is regular of degree k, and

$$k = b_0, \quad b_D = c_0 = 0, \quad c_1 = 1, \quad a_i + b_i + c_i = k.$$



The equilibrium measure of a distance–regular graph

Lemma

Let Γ be a distance–regular graph. Then, for all $y \in V$

$$\nu^x(y) = \sum_{j=0}^{d(x,y)-1} \frac{n - |B_j|}{|\partial B_j|}$$

$$\mathrm{cap}(x) = \sum_{j=0}^{D-1} \frac{(n - |B_j|)^2}{|\partial B_j|}$$

where $|B_j|$ is the number of vertices at distance at most j from a given vertex and $|\partial B_j| = k_j b_j$.

Theorem

The Moore–Penrose inverse of L *is an M–matrix if, and only if, for any* $x \in V$

$$\mathrm{cap}(x) \leq n\nu^x(y) \quad \text{for any } y \sim x.$$

Proof

The Green function is given by

$$G(x,y) = \frac{1}{n^2}\big(\mathrm{cap}(x) - n\,\nu^x(y)\big),$$

But, $\min\limits_{y\in V\setminus\{x\}}\{\nu^x(y)\} = \min\limits_{y\sim x}\{\nu^x(y)\}$, since if the minimum is attained at $z \not\sim x$,

$$1 = \mathcal{L}(\nu^x)(z) = \sum_{y\in V} c(x,y)\big(\nu^x(z) - \nu^x(y)\big) \leq 0.!!$$

The Moore–Penrose inverse of L

Proposition

The Moore–Penrose inverse of L *is an M–matrix if, and only if,*

$$\sum_{i=1}^{D-1} \frac{(n-|B_i|)^2}{|\partial B_i|} \leq \frac{n-1}{k}.$$

Strongly regular graph

For a strongly regular graph with parameters (n, k, a_1, c_2), the Moore–Penrose inverse of L is an M–matrix if, and only if,

$$a_1 \leq 3k - \frac{k^2}{n-1} - n.$$

The n-Cycle

$$\nu^{x_i}(x_j) = \frac{1}{2}\,|i-j|\big(n-|i-j|\big)$$

$$\mathrm{cap}(x_i) = \frac{n(n^2-1)}{12}, \quad i,j = 1,\dots,n.$$

The Moore–Penrose inverse of the combinatorial Laplacian of a n–cycle is a M–matrix if, and only if,

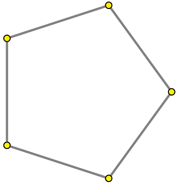
$$\frac{n(n^2-1)}{12} \le \frac{n(n-1)}{2};$$

that is if, and only if, $n \le 5$.
The Moore–Penrose inverse of L is $\mathsf{L}^\dagger = (g_{ij})$ where

$$g_{ij} = \frac{1}{12n}\Big(n^2 - 1 - 6|i-j|(n-|i-j|)\Big), \qquad i,j = 1,\dots,n.$$

5-cycle

The Moore–Penrose inverse of L is an M–matrix



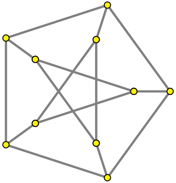
$$\mathsf{L}^{\dagger} = (g_{ij}) = \begin{pmatrix} 2/5 & 0 & -1/5 & -1/5 & 0 \\ 0 & 2/5 & 0 & -1/5 & -1/5 \\ -1/5 & 0 & 2/5 & 0 & -1/5 \\ -1/5 & -1/5 & 0 & 2/5 & 0 \\ 0 & -1/5 & -1/5 & 0 & 2/5 \end{pmatrix}$$

Petersen

The Moore–Penrose inverse of the Laplacian matrix of the Petersen graph is

Petersen

The Moore–Penrose inverse of the Laplacian matrix of the Petersen graph is **NOT** an M–matrix



$$\begin{cases} \nu^x(y) = 3 & \text{if } d(x,y) = 1 \\ \nu^x(y) = 4 & \text{if } d(x,y) = 2 \end{cases}$$

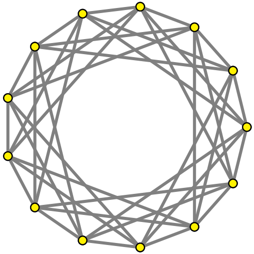
$\text{cap}(x) = 33.$

The Green function of the Petersen Graph is

$$\begin{cases} G(x,x) = 0.33 & \\ G(x,y) = 0.03 & \text{if } d(x,y) = 1 \\ G(x,y) = -0.07 & \text{if } d(x,y) = 2 \end{cases}$$

Conference graphs

The Moore–Penrose inverse of the Laplacian matrix of a conference graph is an M–matrix.



$$\left(n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4}\right)$$

All the best...and thank you!